Strongest Column by the Finite Element Displacement Method

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Theme

AN exhaustive search of the existing literature reveals that the problem of optimization of columns with elastic restraints and subjected to a varying axial load distribution has not received any attention.

In this paper a finite element displacement formulation is used to determine the distribution of material which will maximize the critical load parameter, λ_{cr}, of an Euler-Bernoulli column of specified length and volume, under various boundary conditions (mixed or not—with or without springs) and subject to the constraint that the cross-sectional area is no smaller than a specified value A_0.

Contents

Consideration is restricted to those columns for which the minimum cross-sectional moment of inertia, $I(x)$, and area, $A(x)$, are related by $I(x) = \rho A^n(x)$ (ρ and n are positive constants). This assumption is a restriction but with a suitable choice of ρ and n it covers a large class of structural configurations.

For such a column, shown in Fig. 1, the Rayleigh quotient, λ (the factor by which the given axial load distribution has to be scaled in order to produce instability in the column), is given by

$$\lambda = \left[\int_0^L E\rho A^n(x)(w'')^2\,dx + U_s\right] \Big/ \int_0^L S_0(x)(w')^2\,dx \qquad (1)$$

where λ is stationary with respect to w and

$$U_s = k_T{}^0 w_{|0}{}^2 + k_T{}^L w_{|L}{}^2 + k_R{}^0 (w')_{|0}{}^2 + k_R{}^L (w')_{|L}{}^2$$

$$S_0(x) = P^0 - \int_0^x s(\xi)\,d\xi$$

Next, it is required to maximize λ_{cr} (lowest λ) with respect to variations in the cross-sectional area $A(x)$ subject to the constraint of constant volume V. The new functional λ^*, which must be extremized, is given by

$$\lambda^* = \lambda - \lambda_1\left[\int_0^L A\,dx - V\right]$$

where λ_1 is an undetermined Lagrange multiplier. The requirement that λ^* be stationary with respect to w leads to the well-known stability equation and the associated boundary conditions. Setting the variation of λ^* with respect to A equal to zero leads to the optimality condition

$$A^{n-1}w''^2 = c^2 = \text{const} \qquad (2)$$

Equation (2) is valid only in those regions where $A(x)$ is not prescribed. If A, as determined by the use of Eq. (2), happens to be less than A_0 then the constraint $A = A_0$ has to be satisfied (see Ref. 1).

It can be very easily seen that the optimality condition, Eq. (2), can be expressed in terms of the linear strain energy density per unit area and the average strain energy density as

$$W/A = U/V = E\rho c^2/2 = \text{const} \qquad (3)$$

Note that this condition is independent of n.

Next, the finite element displacement method is used to solve the two variational problems, namely the stability analysis and the satisfaction of the optimality condition. Thus, if $[K]$ and $[K_G]$ denote the assembled nonsingular stiffness and stability matrices, respectively, of the supported column and $\{q\}$ the vector of the unrestrained degrees of freedom of the column, the first variational problem reduces to

$$\big[[K] - \lambda[K_G]\big]\{q\} = \{0\} \qquad (4)$$

In the case of finite elements, the optimality condition, Eq. (3), requires that the strain energy density U_i/v_i of the ith element be equal to the average strain energy density U/V (a constant).

The optimality condition is met by successive iterations starting from a column of uniform cross section complying with the given volume V. Each iteration involves the solution of Eq. (4) to obtain λ_{cr} and $\{q\}_{cr}$. Having obtained these, the strain energy density U_i/v_i in each of the m elements can be calculated as

$$U_i/v_i = \tfrac{1}{2}\{q_i\}^T[k_i]\{q_i\}/[(I_i/\rho)^{1/n} l_i] \qquad (5)$$

where $[k_i]$ is the stiffness matrix of the ith element. This distribution of strain energy density is utilized to decide the inertias of the elements in the next iteration. The recurrence relation for doing so is assumed to be

$$I_i^{r+1} = b^{r+1} I_i^r (U_i^r V / v_i^r U)^p \qquad (6)$$

where the superscripts r and $r+1$ refer to the rth and $(r+1)$st iterations, b^{r+1} is a constant to be determined from the constant volume constraint and the exponent p is assumed to be positive. By the use of Rayleigh's principle it can then be shown that, as long as the ratio inside the parentheses in Eq. (6) is different from unity, a value $p > 0$ exists such that $\lambda_{cr}{}^{r+1} \geq \lambda_{cr}{}^r$. The iterative scheme can be started with a value of p equal to

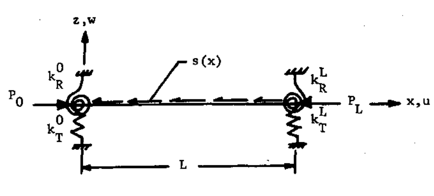

Fig. 1 A typical column with elastically restrained ends subjected to a varying axial load.

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Index categories: Optimal Structural Design; Structural Stability Analysis.

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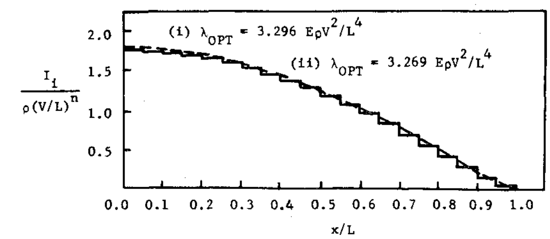

Fig. 2 Optimum moment of inertia distribution for a column with $k_T^0 = \infty$, $k_R^0 = \infty$, $k_T^L = 0$, $k_R^L = 0$; $m = 20$; $n = 2$; (i) ---- exact solution; (ii) —— finite element solution.

$n/n+1$ or less and the scheme can be continued with this value of p as long as $\lambda_{cr}^{r+1} \geqq \lambda_{cr}^{r}$. If $\lambda_{cr}^{r+1} < \lambda_{cr}^{r}$ then the value of p is reduced by a factor of $\frac{1}{2}$ or $\frac{1}{4}$ and the iteration is repeated. This process is carried on till no substantial change either in the value of λ_{cr} or in the value of the moment of inertia I_i of each element is possible and the strain energy density distribution is essentially uniform.

In the case of the inequality constraint, the optimization procedure remains exactly the same as the one just outlined until such time at which the inertias of some elements violate the inequality constraint. The inertias of those elements are set equal to the prescribed value ρA_0^n while the inertias of the remaining elements are adjusted to satisfy the volume constraint.

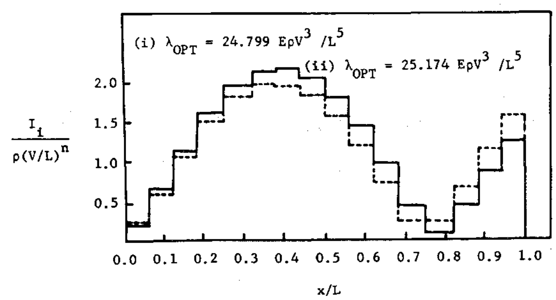

Fig. 3 Optimum moment of inertia distribution for a column with $k_T^0 = \infty$, $k_R^0 = 0$, $k_T^L = \infty$, $k_R^L = 25E\rho V^3/L^4$; $m = 16$; $n = 3$; (I) ---- constrained $I \geqq 0.25\rho V^3/L^3$; (ii) —— unconstrained.

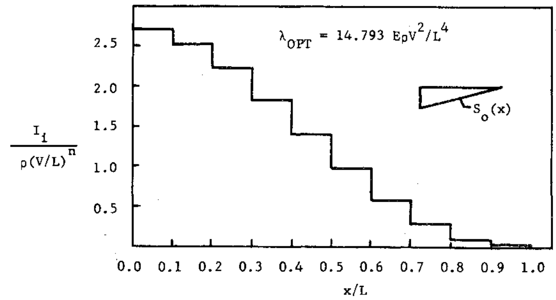

Fig. 4 Optimum moment of inertia distribution for a column with $k_T^0 = \infty$, $k_R^0 = \infty$, $k_T^L = 0$, $k_R^L = 0$; $S_0(x) = E\rho V^2(L-x)/L^5$; $m = 10$; $n = 2$.

The optimality condition of constant strain energy density will naturally be satisfied only in those regions where the inequality constraint is not effective.

The criterion for convergence on the optimality condition is $[(U_i/v_i)_{max}/(U_j/v_j)_{min} - 1.0] \leqq 0.005$. Results are presented graphically for some typical boundary conditions and destabilizing loads in Figs. 2–4 (see Refs. 2 and 3 for greater details).

In general, through the use of the finite element displacement method, optimization of columns under all possible boundary conditions and destabilizing loads can be successfuly accomplished. It can be seen that, although ρ is assumed to be a constant, cases for which ρ is a function of x or for which n takes on different values over different portions of the column can be treated similarly. In addition, the effect of shear deformation can be accounted for through a change in the element stiffness matrix to include this effect. It is of course clear that this treatment does not include the optimal design of columns for multiple loadings.

References

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